Let f, g be T-periodic, continuous and piecewise C^1.

Q: $$\int_0^T |f(x)|^2 dx \lesssim \int_0^T |f'(x)|^2 dx \quad (*) \ ?$$

$$\left|\int_0^T \overline{f(x)} g(x)\, dx\right|^2 \lesssim \int_0^T |f'(x)|^2 dx \int_0^T |g'(x)|^2 dx \quad (**) \ ?$$

Recall Parseval's Identity:

$$\frac{1}{T}\int_0^T |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

where $\hat{f}(n) = \frac{1}{T}\int_0^T f(x) e^{-2\pi i n x/T} dx$

I. $\hat{f}(0) := \frac{1}{T}\int_0^T f(x)\, dx = 0 \Rightarrow (*)$ holds

Proof: Note that for $n \neq 0$

$$\hat{f'}(n) = \frac{1}{T}\int_0^T f'(x) e^{-2\pi i n x/T} dx$$

Integral by part $$= \frac{f(x) e^{-2\pi i n x/T}}{T}\Bigg|_0^T - \left(\frac{-2\pi i n/T}{T}\right)\int_0^T f(x) e^{-2\pi i n x/T} dx$$

$$= \frac{2\pi i n}{T} \hat{f}(n)$$

$$\hat{f'}(0) = \frac{1}{T}\int_0^T f'(x)\,dx = \frac{1}{T} f(x)\Big|_0^T = 0$$

Fundamental Theorem of Calculus

T-periodic

$$\frac{1}{T}\int |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$ Parseval's Identity

$$= \left(\frac{T}{2\pi n}\right)^2 \sum_{n=-\infty}^{\infty} |\hat{f'}(n)|^2$$ $\hat{f}(0) = 0$

$$\leq \left(\frac{T}{2\pi}\right)^2 \sum_{n=-\infty}^{\infty} |f'(n)|^2$$

$$= \left(\frac{T}{2\pi}\right)^2 \frac{1}{T}\int_0^T |f'(x)|^2\,dx$$ Parseval's Identity

$$= \frac{T}{4\pi^2}\int_0^T |f'(x)|^2\,dx$$

$\square$

II. $\hat{f}(0) = 0 \Rightarrow (**)$ holds

Rmk: If $\hat{f}(0) = \hat{g}(0) = 0$, then

$$\left| \int_0^T \overline{f(x)} g(x) \, dx \right|^2 \leq \int_0^T |f(x)|^2 dx \int_0^T |g(x)|^2 dx$$ Cauchy-Schwartz

$$\leq \frac{T^2}{4\pi^2} \int_0^T |f'(x)|^2 dx \int_0^T |g'(x)|^2 dx$$ (I)

Now we only assume $\hat{f}(0)=0$ with $\hat{g}(0) \neq 0$.

$(**)$ still holds.

Proof: Let $h(x) = g(x) - \hat{g}(0)$.

Then $\hat{h}(0) = \hat{g}(0) - \hat{g}(0) = 0$ and $h'(x) = g'(x)$

Then $\left| \int_0^T \overline{f(x)} g(x) \, dx \right|^2$

$$= \left| \int_0^T \overline{f(x)} g(x) \, dx - \left(\int_0^T \overline{f(x)} \, dx \right) \hat{g}(0) \right|^2$$ $\hat{f}(0)=0$

$$= \left| \int_0^T \overline{f(x)} h(x) \, dx \right|^2$$ $h(x) = g(x) - \hat{g}(0)$

$$\leq \frac{T^2}{4\pi^2} \int |f'(x)|^2 dx \int |h'(x)|^2 dx$$ by Rmk

$$= \frac{T^2}{4\pi^2} \int |f'(x)|^2 dx \int |g'(x)|^2 dx$$ $h'(x) = g'(x)$

$\square$

III. $f(0)=f(T)=0 \implies (*)$ holds.

Rmk: This assumption guarantees the continuity of the odd extension so that we can apply FTC.

Proof: Let $h(x)=\begin{cases} f(x), & x\in[0,T] \\ -f(-x), & x\in[-T,0) \end{cases}$ and

extend h to a $2T$-periodic function.

Since $f(0)=f(T)$, h is continuous.

Also $\displaystyle\int_0^{2T} h(x)\,dx \underset{h \text{ is } 2T\text{-periodic}}{=} \int_{-T}^{T} h(x)\,dx \underset{\text{odd}}{=} 0.$

Then $\displaystyle\int_0^T |f(x)|^2dx = \frac{1}{2}\int_0^{2T}|f(x)|^2dx$ $\quad f$ is T-periodic

$\displaystyle = \frac{1}{2}\int_0^{2T}|h(x)|^2dx$

$\displaystyle \le \frac{1}{2}\frac{T^2}{\pi^2}\int_0^{2T}|h'(x)|^2dx$ $\quad (I)$

$\displaystyle = \frac{1}{2}\frac{T^2}{\pi^2}\int_0^{2T}|f'(x)|^2dx$

$\displaystyle = \frac{T^2}{\pi^2}\int_0^{T}|f'(x)|^2dx$ $\quad f'$ is T-periodic

$\square$